M.Math. IInd Year
IInd Semester backpaper Exam 2011
Algebraic Number Theory
Instructor : B.Sury
Attempt any 5 questions.

1. Show that in any algebraic number field $K \neq Q$, some prime number p must ramify.

2. For a prime p, prove that the splitting field of $1+X+X^2+\cdots+X^{p-1}$ over Q_p is $Q_p\left((-p)^{\frac{1}{p-1}}\right)$.

3. Let k be a finite extension of Q_p and $(O, \mathcal{P})$ be the corresponding local ring in k. Prove that $\exists\ c, d$ so that the p-th power map is an isomorphism from $1+\mathcal{P}^i$ to $1+\mathcal{P}^{i+d}$ $\forall\ i > c$. If k contains nontrivial p-th roots of unity, find c such that the p-th power map from $1+\mathcal{P}^c$ to $1+\mathcal{P}^{c+d}$ has a kernel $\neq \{1\}$.

4. Let $K = Q(\zeta)$, where ζ is a primitive n-th root of unity. If S is the set of prime ideals containing (n) and $I_Q(S)$ denotes the group of fractional ideals generated by primes outside S, find (with proof) the image and the kernel of the Artin map from $I_Q(S)$ to Gal (K/Q).

5. State the Frobenius density theorem and use it to prove that in a cyclic extension of algebraic number fields, infinitely many primes are inert.

6. Describe all finite, unramified extensions of $\mathbf{Q}_p$.

7. Let L/K be a cyclic extension of algebraic number fields. For a finite place v of L, show that the unit group O_v^* of the local ring O_v of integers in the completion L_v, is a module for the decomposition group at v and that $H^1(O_v^*)$ has order e_v.

8. Prove the Artin reciprocity theorem for abelian extensions assuming it time for cyclic extensions.

9. Let L/K be cyclic, $i : K^* \to I_K$ the map $\alpha \mapsto (\alpha)$. Suppose c is a modulus of K such that $K_{c,1} \cap i^{-1}(N(I_L(c)))$ is contained in $N(L^*)$. Show that elements of K^* which are local norms at all primes dividing c, are global norms.